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Published in:
Journal fuer die Reine und Angewandte Mathematik

DOI:
[10.1515/crelle-2014-0005](https://doi.org/10.1515/crelle-2014-0005)

Publication date:
2016

Document version
Peer reviewed version

Document license:
[Unspecified](#)

Citation for published version (APA):
Ando, H., Haagerup, U., & Winsløw, C. (2016). Ultraproducts, QWEP von Neumann algebras, and the Effros–Maréchal topology. *Journal fuer die Reine und Angewandte Mathematik*, 715, 231-250.
<https://doi.org/10.1515/crelle-2014-0005>

Ultraproducts, QWEP von Neumann Algebras, and the Effros-Maréchal Topology

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May 13, 2013

Abstract

Based on the analysis on the Ocneanu/Groh-Raynaud ultraproducts and the Effros-Maréchal topology on the space $\text{vN}(H)$ of von Neumann algebras acting on a separable Hilbert space H , we show that for a von Neumann algebra $M \in \text{vN}(H)$, the following conditions are equivalent: (1) M has the Kirchberg's quotient weak expectation property (QWEP). (2) M is in the closure of the set \mathcal{F}_{inj} of injective factors on H with respect to the Effros-Maréchal topology. (3) M admits an embedding i into the Ocneanu ultrapower R_∞^ω of the injective III_1 factor with a normal faithful conditional expectation $\varepsilon: R_\infty^\omega \rightarrow i(M)$. (4) For every $\varepsilon > 0$, $n \in \mathbb{N}$ and $\xi_1, \dots, \xi_n \in \mathcal{P}_M^b$, there is $k \in \mathbb{N}$ and $a_1, \dots, a_n \in M_k(\mathbb{C})_+$, such that $|\langle \xi_i, \xi_j \rangle - \text{tr}_k(a_i a_j)| < \varepsilon$ ($1 \leq i, j \leq n$), where tr_k is the tracial state on $M_k(\mathbb{C})$, and \mathcal{P}_M^b is the natural cone in the standard form of M .

1 Introduction

In the seminal paper [Kir93], Kirchberg revealed the unexpected connection among tensor products of C^* -algebras, the weak expectation property (WEP) of Lance [La73], and the Connes's embedding problem type II_1 factors. A C^* -algebra is said to have WEP, if for any faithful representation $A \subset \mathbb{B}(H)$, there is a unital completely positive (ucp) map $\Phi: \mathbb{B}(H) \rightarrow A^{**}$ such that $\Phi|_A = \text{id}_A$. A C^* -algebra A is said to have the quotient weak expectation property (QWEP) if there is a surjective $*$ -homomorphism from a C^* -algebra B with WEP onto A . Among other interesting results, Kirchberg proved that the following conditions are equivalent:

- (1) $C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$.
- (2) Every C^* -algebra has QWEP.
- (3) Every II_1 factor N with separable predual embeds into the tracial ultrapower R^ω (for some $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$) of the hyperfinite II_1 factor R .

(2) is now called Kirchberg's QWEP conjecture, and (3) is called the Connes's embedding conjecture. It was mentioned for the first time in [?]. See also the excellent survey [Oz04] on the QWEP conjecture. Nowadays many interesting equivalent conditions of the above conjectures are known. On the other hand,

the QWEP conjecture also has another connection to the topological properties of the space of von Neumann algebras. The Effros-Maréchal topology on the space $\mathbf{vN}(H)$ of von Neumann algebras acting on a fixed Hilbert space H is the weakest topology on $\mathbf{vN}(H)$ for which the map

$$N \mapsto \|\varphi|_N\|$$

is continuous for every $\varphi \in \mathbb{B}(H)_*$. Based on the work of Effros [Eff65-1, Eff65-2] on the Borel structure of $\mathbf{vN}(H)$, this topology was defined by Maréchal in [Mar73] so that it generates the Effros Borel structure. Later it was intensively studied in [HW98, HW00], where the second and third-named authors showed that this topology was indeed well-matched with Tomita-Takesaki theory and could be used as a tool for the study of global properties of von Neumann algebras. Among other things, it was proved [HW00, Theorem 5.8] that when H is separable, a II_1 factor $N \in \mathbf{vN}(H)$ is in the closure of the set \mathcal{F}_{inj} of injective factors on H , if and only if N embeds into R^ω . Consequently, Connes's embedding conjecture (hence all conditions (1)-(3) above) is equivalent to

- (4) \mathcal{F}_{inj} is dense in $\mathbf{vN}(H)$.

In this paper, we establish further connection among ultraproducts, the approximation by injective factors, and QWEP von Neumann algebras. To this we make use of the recent work of the first and the second-named authors on the ultraproducts of general von Neumann algebras [AH12]. We also carry on the analysis of the natural cone in the standard form [Haa75]. The main result of the paper is as follows. Let R_∞ (resp. R_λ) denote the injective factor of type III_1 (resp. type III_λ ($0 < \lambda < 1$)). We assume H is separable infinite-dimensional.

Theorem 1.1. *Let $0 < \lambda < 1$ and let $M \in \mathbf{vN}(H)$, and let \mathcal{P}_M^\natural be the natural cone in the standard form of M . The following conditions are equivalent.*

- (1) M has QWEP.
- (2) $M \in \overline{\mathcal{F}_{\text{inj}}}$.
- (3) *There is an embedding $i: M \rightarrow R_\infty^\omega$ and a normal faithful conditional expectation $\varepsilon: R_\infty^\omega \rightarrow i(M)$.*
- (4) *There is an embedding $i: M \rightarrow R_\lambda^\omega$ and a normal faithful conditional expectation $\varepsilon: R_\lambda^\omega \rightarrow i(M)$.*
- (5) *There is $\{k_n\}_{n=1}^\infty \subset \mathbb{N}$ and a normal faithful state φ_n on $M_{k_n}(\mathbb{C})$ ($n \in \mathbb{N}$), an embedding $i: M \rightarrow (M_{k_n}(\mathbb{C}), \varphi_n)^\omega$ and a normal faithful expectation $\varepsilon: (M_{k_n}(\mathbb{C}), \varphi_n)^\omega \rightarrow i(M)$.*
- (6) *For every $\varepsilon > 0, n \in \mathbb{N}$ and $\xi_1, \dots, \xi_n \in \mathcal{P}_M^\natural$, there exist $k \in \mathbb{N}$ and $a_1, \dots, a_n \in M_k(\mathbb{C})_+$ such that*

$$|\langle \xi_i, \xi_j \rangle - \text{tr}_k(a_i a_j)| < \varepsilon \quad (1 \leq i, j \leq n).$$

Here, $(N_n, \varphi_n)^\omega$ denotes the Ocneanu ultraproduct of $\{N_n\}_{n=1}^\infty$ with respect to the sequence of normal faithful states $\{\varphi_n\}_{n=1}^\infty$ (see §2).

The organization of the paper is as follows. In §2 we recall necessary backgrounds about ultraproducts, Effros-Maréchal topology. In §3, we prove the

relation between embedding into the Ocneanu ultraproduct and the limit in the Effros-Maréchal topology. Using this, we prove the equivalence of (1)-(5) in the above theorem. In §4, we prove the equivalence of (6) and (1) in the above theorem.

2 Preliminaries

2.1 Notation

Throughout the paper, $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ denotes a fixed free ultrafilter on \mathbb{N} . H denotes a separable Hilbert space and $\text{vN}(H)$ the space of all von Neumann algebras acting on H (we assumed all von Neumann algebras contain $1 = \text{id}_H$). Let M be a von Neumann algebra, φ be a normal state on M . As usual, we define two norms $\|\cdot\|_\varphi$, $\|\cdot\|_\varphi^\sharp$ by

$$\|x\|_\varphi := \varphi(x^*x)^{\frac{1}{2}}, \quad \|x\|_\varphi^\sharp := \varphi(x^*x + xx^*)^{\frac{1}{2}}, \quad x \in M.$$

For a sequence $(M_n)_n$ of von Neumann algebras, $\ell^\infty(\mathbb{N}, M_n)$ is the C*-algebra of all norm-bounded sequences $(x_n)_n \in \prod_{n \in \mathbb{N}} M_n$.

2.2 Natural cone \mathcal{P}_M^\natural

Recall the following [Haa75]:

Definition 2.1. Let $(M, H, J, \mathcal{P}_M^\natural)$ be a quadruple, where M is a von Neumann algebra, H is a Hilbert space on which M acts, J is an antilinear isometry on H with $J^2 = 1$, and $\mathcal{P}_M^\natural \subset H$ is a closed convex cone which is self-dual, i.e., $\mathcal{P}_M^\natural = (\mathcal{P}_M^\natural)^0$, where

$$(\mathcal{P}_M^\natural)^0 := \{\xi \in H; \langle \xi, \eta \rangle \geq 0, \eta \in \mathcal{P}_M^\natural\}.$$

Then $(M, H, J, \mathcal{P}_M^\natural)$ is called a *standard form* if the following conditions are satisfied:

1. $JMJ = M'$.
2. $J\xi = \xi$, $\xi \in \mathcal{P}_M^\natural$.
3. $xJxJ(\mathcal{P}_M^\natural) \subset \mathcal{P}_M^\natural$, $x \in M$.
4. $JxJ = x^*$, $x \in \mathcal{Z}(M)$.

We remark that condition 4. automatically follows from the other three conditions [AH12, Lemma 3.19]. The existence and the uniqueness of the standard form was established in [Haa75]. \mathcal{P}_M^\natural is called the *natural cone* of M . It was independently introduced by Connes [Con72] and Araki [Ara74], and if M is σ -finite with a normal faithful state φ , then on the GNS Hilbert space $H = L^2(M, \varphi)$, \mathcal{P}_M^\natural is realized as

$$\mathcal{P}_M^\natural = \overline{\{aJ_\varphi aJ_\varphi \xi_\varphi; a \in M\}} = \overline{\{\Delta_\varphi^{\frac{1}{4}} a \xi_\varphi; a \in M_+\}}.$$

Here, Δ_φ (resp. J_φ) is the modular (resp. modular conjugation) operator of φ . \mathcal{P}_M induces an order structure on H by

$$\xi \leq \eta \Leftrightarrow \xi - \eta \in \mathcal{P}_M^h.$$

This order structure was intensively studied in [Con74, Ara74]. Among others, it holds that [Ara74] every positive $\varphi \in M_*$ is represented as $\varphi = \omega_{\xi_\varphi}$ by a unique vector $\xi_\varphi \in \mathcal{P}_M^h$ and we have the Araki-Powers-Størmer inequality:

$$\|\xi_\varphi - \xi_\psi\|^2 \leq \|\varphi - \psi\| \leq \|\xi_\varphi - \xi_\psi\| \|\xi_\varphi + \xi_\psi\|.$$

The readers are referred to these papers for more detailed information on the natural cone.

2.3 The Ocneanu and the Groh-Raynaud Ultraproduct

Let $(M_n, \varphi_n)_n$ be a sequence of pairs of σ -finite von Neumann algebras equipped with normal faithful states. Let $\mathcal{I}_\omega := \mathcal{L}_\omega \cap \mathcal{L}_\omega^*$, where

$$\mathcal{L}_\omega = \mathcal{L}_\omega(M_n, \varphi_n) := \{(x_n)_n \in \ell^\infty(\mathbb{N}, M_n); \|x_n\|_{\varphi_n} \xrightarrow{n \rightarrow \omega} 0\},$$

and let \mathcal{M}^ω be the multiplier algebra of \mathcal{I}_ω . The *Ocneanu ultraproduct* $(M_n, \varphi_n)^\omega$ of $(M_n, \varphi_n)_n$ is the quotient algebra $\mathcal{M}^\omega / \mathcal{I}_\omega$. For each $n \in \mathbb{N}$, let $H_n = L^2(M_n, \varphi_n)$ be the GNS Hilbert space of (M_n, φ_n) , and let H_ω be the ultraproduct of $(H_n)_n$. Define a $*$ -representation $\pi_\omega : \ell^\infty(\mathbb{N}, M_n) \rightarrow \mathbb{B}(H_\omega)$ by

$$\pi_\omega((x_n)_n)(\xi_n)_\omega := (x_n \xi_n)_\omega, \quad (x_n)_n \in \ell^\infty(\mathbb{N}, M_n), (\xi_n)_\omega \in H_\omega.$$

The *Groh-Raynaud ultraproduct* $\prod^\omega M_n$ is the von Neumann algebra $\pi_\omega(\ell^\infty(\mathbb{N}, M_n))''$ acting on H_ω . Let $\xi_n \in H_n$ be the cyclic vector corresponding to φ_n , and let $\xi_\omega = (\xi_n)_\omega \in H_\omega$. Let $p \in \prod^\omega M_n$ be the support projection of the normal state $\langle \cdot, \xi_\omega, \xi_\omega \rangle$ on $\prod^\omega M_n$. Then by [AH12, Proposition 3.15], $(M_n, \varphi_n)^\omega \cong p(\prod^\omega M_n)p$. For more details about ultraproducts, see [AH12].

2.4 Effros-Maréchal Topology on $\text{vN}(H)$

As explained in the introduction, the *Effros-Maréchal topology* is the weakest topology on $\text{vN}(H)$ which makes each functional $\text{vN}(H) \ni N \mapsto \|\varphi|_N\|$ continuous ($\varphi \in \mathbb{B}(H)_*$). As H is separable, this makes $\text{vN}(H)$ a Polish space, and the topology can be described by the concepts of \limsup and \liminf of a sequence in $\text{vN}(H)$ (the \limsup presented here is a refinement [HW98] of s - \limsup given in [Tuk85]). Since we only consider the case where H is separable, we use the following simplified equivalent definitions (see [HW98, §2] for the original definition).

Definition 2.2. Let $(M_n)_n \subset \text{vN}(H)$.

- (1) $\liminf M_n$ is the set of all $x \in \mathbb{B}(H)$ for which there exists $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ that converges to x $*$ -strongly.
- (2) $\limsup M_n$ is the von Neumann algebra generated by the set of all $x \in \mathbb{B}(H)$ for which there exists $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ such that x is a weak-limit point of $\{x_n\}_{n=1}^\infty$.

It is proved in [HW98, Theorems 2.8, 3.5] that for a sequence $\{M_n\}_{n=1}^\infty \subset \mathbf{vN}(H)$ and $M \in \mathbf{vN}(H)$,

$$\lim_{n \rightarrow \infty} M_n = M \text{ in } \mathbf{vN}(H) \Leftrightarrow \liminf_{n \rightarrow \infty} M_n = M = \limsup_{n \rightarrow \infty} M_n,$$

$$\left(\limsup_{n \rightarrow \infty} M_n \right)' = \liminf_{n \rightarrow \infty} M_n'.$$

We will also make use of the ultralimit version of the above concepts.

Definition 2.3. Let $(M_n)_n \subset \mathbf{vN}(H)$. We define

- (1) $\liminf_{n \rightarrow \omega} M_n$ is the set of all $x \in \mathbb{B}(H)$ for which there exists $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ such that $x = \text{so}^* - \lim_{n \rightarrow \omega} x_n$.
- (2) $\limsup_{n \rightarrow \omega} M_n$ is the von Neumann algebra generated by the set of all $x \in \mathbb{B}(H)$ for which there exists $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ such that $x = \text{wo} - \lim_{n \rightarrow \omega} x_n$.

Again they are different from the original definitions. It also holds that [HW00, Lemma 5.2]

$$M = \lim_{n \rightarrow \omega} M_n \text{ in } \mathbf{vN}(H) \Leftrightarrow \liminf_{n \rightarrow \omega} M_n = M = \limsup_{n \rightarrow \omega} M_n,$$

$$\left(\limsup_{n \rightarrow \omega} M_n \right)' = \liminf_{n \rightarrow \omega} M_n'.$$

Finally, we will consider the following subsets of $\mathbf{vN}(H)$ (see the introduction of [HW00] for more details):

- \mathcal{F} , the set of factors acting on H .
- \mathcal{F}_X , the set of factors of type X acting on H , where X runs among the standard numberings of the types of factors, such as II_1 or III_λ .
- \mathcal{F}_{inj} , the set of injective factors acting on H .
- \mathcal{F}^{st} , the set of factors acting standardly on H .
- $\text{SA}(M)$ the set of von Neumann subalgebras of $M \in \mathbf{vN}(H)$.

For a fixed normal faithful state φ on M , we denote $\text{SA}_\varphi(M)$ to be the set of those $N \in \text{SA}(M)$ satisfying $\sigma_t^\varphi(N) = N$ ($t \in \mathbb{R}$) (see [HW98, §2]). We also make use of self-explanatory extensions of the above notation, such as $\mathcal{F}_{\text{inj}}^{\text{st}} = \mathcal{F}_{\text{inj}} \cap \mathcal{F}^{\text{st}}$.

3 Embedding into the Ocneanu Ultraproducts and Effros-Maréchal Topology

In this section, we consider a separable infinite-dimensional Hilbert space H . An *embedding* is understood to be a unital normal injective $*$ -homomorphism of one von Neumann algebra into another. The main technical part of this section deals with versions of the following statements about von Neumann algebras $M, N \in \mathbf{vN}(H)$:

1. There is a sequence $(\psi_n)_n \subset S_{\text{nf}}(M)$, an embedding of $i: N \rightarrow (M, \psi_n)^\omega$ and a faithful normal conditional expectation ε of $(M, \psi_n)^\omega$ onto $i(N)$;
2. There is a separable Hilbert space K , a sequence of isomorphic copies $M_n \in \text{vN}(K)$ of M and an isomorphic copy $N_0 \in \text{vN}(K)$ of N such that $M_n \rightarrow N_0$ in the Effros-Maréchal topology;

We prove that 1. is equivalent to 2. (Theorem 3.1 and Theorem 3.3) As an application we give a new characterization of the closure of the set of injective factors in $\text{vN}(H)$ (see [HW00, §4-5] for more details).

3.1 Approximation Theorem and The Closure of Injective Factors.

We begin by the following

Theorem 3.1. *Let $N \in \text{vN}(H)$. Assume we are given a sequence $(M_n) \subset \text{vN}(H)$ such that $M_n \rightarrow N$ in $\text{vN}(H)$. Fix $\chi \in S_{\text{nf}}(\mathbb{B}(H))$, and let $\psi_n := \chi|_{M_n}$, $\varphi := \chi|_N$. Then there exists an embedding i of N into $(M_n, \psi_n)^\omega$ such that $\varphi = (\psi_n)^\omega \circ i$, and a normal faithful conditional expectation ε of $(M_n, \psi_n)^\omega$ onto $i(N)$ such that $(\psi_n)^\omega \circ \varepsilon = (\psi_n)^\omega$.*

Proof. Let $x \in N$. By [HW98, Theorem 2.8], $N = \liminf_{n \rightarrow \infty} M_n$ and we may choose $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ satisfying $x_n \xrightarrow{\text{so}^*} x$. It is clear that $(x_n)_n \in \mathcal{M}^\omega(M_n, \psi_n)$ because of the joint continuity of operator product in strong*-topology on bounded sets, and also that $(x_n)^\omega$ is independent of the choice of the sequence $(x_n)_n$ converging to x . Hence we may define $i: N \rightarrow (M_n, \psi_n)^\omega$ by $i(x) = (x_n)^\omega$, which is clearly an injective *-homomorphism, and as

$$(\psi_n)^\omega \circ i(x) = \lim_{n \rightarrow \omega} \psi_n(x_n) = \lim_{n \rightarrow \omega} \chi(x_n) = \varphi(x),$$

we have $(\psi_n)^\omega \circ i = \varphi$. In particular, i is normal. Next, for $y = (x_n)^\omega \in (M_n, \psi_n)^\omega$, let $x := \text{wo-}\lim_{n \rightarrow \omega} x_n$. Then $x \in \limsup_{n \rightarrow \infty} M_n = N$ by [HW98, Theorem 2.8]. We then define $\varepsilon(y) := i(x)$. Let $x \in N$ and suppose $i(x) = (x_n)^\omega$ for $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$. Then as $x_n \xrightarrow{\text{so}^*} x$, we have $\varepsilon \circ i(x) = \varepsilon((x_n)^\omega) = i(x)$, whence $\varepsilon \circ i = i$. Therefore ε is a faithful conditional expectation of $(M_n, \psi_n)^\omega$ onto $i(N)$. Next, let $(x_n)^\omega \in (M_n, \psi_n)^\omega$ and let $x = \text{wo-}\lim_{n \rightarrow \omega} x_n$. Suppose $i(x) = (x'_n)^\omega$, where $(x'_n)_n \in \ell^\infty(\mathbb{N}, M_n)$. Then

$$\begin{aligned} (\psi_n)^\omega \circ \varepsilon((x_n)^\omega) &= (\psi_n)^\omega \circ i(x) = (\psi_n)^\omega((x'_n)^\omega) \\ &= \lim_{n \rightarrow \omega} \psi_n(x'_n) = \lim_{n \rightarrow \omega} \chi(x'_n) = \chi(x) \\ &= \lim_{n \rightarrow \omega} \chi(x_n) = (\psi_n)^\omega((x_n)^\omega), \end{aligned}$$

whence $(\psi_n)^\omega \circ \varepsilon = (\psi_n)^\omega$. Since $(\psi_n)^\omega$ is normal and faithful, ε is also normal. \square

Corollary 3.2. *Let $M, N \in \text{vN}(H)$, and assume $(M_n) \subset \text{vN}(H)$ has $M_n \cong M(n \in \mathbb{N})$ and $M_n \rightarrow N$. Then there exists a sequence $(\psi_n)_n \subset S_{\text{nf}}(M)$ and an embedding i of N into $(M, \psi_n)^\omega$, and a normal faithful conditional expectation of $(M, \psi_n)^\omega$ onto $i(N)$.*

Proof. Let $\varphi \in S_{\text{nf}}(N)$. Using [HW98, Lemma 5.6], we find $\chi \in S_{\text{nf}}(\mathbb{B}(H))$ such that $\varphi = \chi|_N$. Let $\psi'_n := \chi|_{M_n} \in S_{\text{nf}}(M_n)$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, take a $*$ -isomorphism $\Phi_n: M \rightarrow M_n$ and let $\psi_n := \psi'_n \circ \Phi_n$. Then $\Phi((x_n)^\omega) = (\Phi_n(x_n))^\omega$ defines a $*$ -isomorphism $\Phi: (M, \psi_n)^\omega \rightarrow (M_n, \psi'_n)^\omega$. By Theorem 3.1, there exists an embedding $i': N \rightarrow (M_n, \psi'_n)^\omega$ and a normal faithful conditional expectation $\varepsilon': (M_n, \psi'_n)^\omega \rightarrow i'(N)$. Then $i := \Phi^{-1} \circ i': N \rightarrow (M, \psi_n)^\omega$ and $\varepsilon := \Phi^{-1} \circ \varepsilon' \circ \Phi: (M, \psi_n)^\omega \rightarrow i(N)$ are the required embedding and the normal faithful conditional expectation. \square

We now prove a partial converse to Theorem 3.1, which is at the same time a generalization of [HW00, Lemma 5.5] to the non-tracial case:

Theorem 3.3. *Let $\{M_n\}_{n=1}^\infty \subset \text{vN}^{\text{st}}(H)$ and $N \in \text{vN}^{\text{st}}(H)$. Let $\psi_n \in S_{\text{nf}}(M_n)$ ($n \in \mathbb{N}$). Assume that we are given an embedding $i: N \rightarrow (M_n, \psi_n)^\omega$ and a normal faithful conditional expectation $\varepsilon: (M_n, \psi_n)^\omega \rightarrow i(N)$. Then there exists $(u_n)_n \subset \mathcal{U}(H)$ and a strictly increasing sequence $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ such that $u_{n_k} M_{n_k} u_{n_k}^* \rightarrow N$ in $\text{vN}(H)$.*

Proof. For simplicity, we use notations $\widetilde{M} = (M, \psi_n)^\omega$ and $\widetilde{\psi} = (\psi_n)^\omega$. Let $\tilde{\varphi} := \widetilde{\psi} \circ \varepsilon \in S_{\text{nf}}(\widetilde{M})$. Then by [AH12, Corollary 3.29], there exists $\varphi_n \in S_{\text{nf}}(M_n)$ ($n \in \mathbb{N}$) such that $(M, \varphi_n)^\omega = \widetilde{M}$ and $\tilde{\varphi} = (\varphi_n)^\omega$. Put $\varphi := \tilde{\varphi}|_{i(N)}$. Since each M_n acts standardly on H , there exists a cyclic and separating vector $\xi_n \in H$ for M_n such that $\varphi_n(x) = \langle x\xi_n, \xi_n \rangle$ ($x \in M_n$) for $n \in \mathbb{N}$. Let $\tilde{\xi} := (\xi_n)_\omega \in H_\omega$. Define as in [AH12, Theorem 3.7] an isometry $w: L^2(\widetilde{M}, \widetilde{\psi}) \rightarrow H_\omega$ given as the unique extension of $(x_n)^\omega \xi_{\tilde{\varphi}} \mapsto (x_n \xi_n)_\omega$ ($(x_n)^\omega \in \widetilde{M}$). Put $K := w(L^2(\widetilde{M}, \tilde{\varphi}))$. Let $J_n: H \rightarrow H$ be the modular conjugation associated with the standard representation of M_n on H . Let $J_\omega := (J_n)_\omega: H_\omega \rightarrow H_\omega$. Then by [AH12, Theorems 3.7 and 3.19], J_ω is the modular conjugation associated with the standard representation of $\prod^\omega M_n$ on H_ω , and $J_\omega|_K = wJ_{\widetilde{M}}w^*$. Here, $J_{\widetilde{M}}$ is the modular conjugation associated with the standard representation of \widetilde{M} on $L^2(\widetilde{M}, \tilde{\varphi})$. Since $\varepsilon(\widetilde{M}) = i(N)$, $\varphi = \tilde{\varphi}|_{i(N)}$ and $\tilde{\varphi} = \varphi \circ \varepsilon$ holds. Thus we may regard $L^2(i(N), \varphi)$ as a closed subspace of $L^2(\widetilde{M}, \tilde{\varphi})$ (thus $\xi_\varphi = \xi_{\tilde{\varphi}}$). In this case by the argument in [Tak72] $J_{i(N)} = J_{\widetilde{M}}|_{L^2(i(N), \varphi)}$ holds, where $J_{i(N)}$ is the modular conjugation associated with the natural representation π_φ of N on $L^2(i(N), \varphi)$ given by $\pi_\varphi(x)i(y)\xi_\varphi = i(xy)\xi_\varphi$ ($x, y \in N$). Put $L := w(L^2(i(N), \varphi)) \subset K \subset H_\omega$ and $w_0 := w|_{L^2(i(N), \varphi)}$. Let P_L (resp. P_K) be the projection of H_ω onto L (resp. K), and let e_L be the projection of K onto L . Let e be the projection of $L^2(\widetilde{M}, \tilde{\varphi})$ onto $L^2(i(N), \varphi)$.

$$\begin{array}{ccccc}
L^2(\widetilde{M}, \tilde{\varphi}) & \xrightarrow{w} & K & \xleftarrow{P_K} & H_\omega \\
\downarrow e & & \downarrow e_L & \nearrow P_L & \\
L^2(i(N), \varphi) & \xrightarrow{w_0} & L & &
\end{array}$$

Since N_* is separable, L is a separable subspace of H_ω . So by [HW00, Lemma 5.1], there exist unitaries $v_n \in \mathcal{U}(L, H)$ ($n \in \mathbb{N}$) such that $\xi = (v_n \xi)_\omega$ holds for every $\xi \in L$. Recall that to construct the Groh-Raynaud ultraproduct $\prod^\omega M_n$, we used $\pi_\omega: \ell^\infty(\mathbb{N}, M_n) \rightarrow \mathbb{B}(H_\omega)$ given by $\pi_\omega((x_n)_n)(\xi_n)_\omega = (x_n \xi_n)_\omega$ for $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ and $(\xi_n)_\omega \in H_\omega$. Note that the definition of π_ω is

slightly different from the one given in [AH12, §3]. Define an analogous map $\hat{\pi}_\omega: \ell^\infty(\mathbb{N}, M'_n) \rightarrow \mathbb{B}(H_\omega)$.

Claim 1. Let $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ and $(x'_n)_n \in \ell^\infty(\mathbb{N}, M'_n)$. Then as elements of $\mathbb{B}(H_\omega)$, the following holds.

$$\text{wo} - \lim_{n \rightarrow \omega} v_n^* x_n v_n = P_L \pi_\omega((x_n)_n) P_L, \quad (1)$$

$$\text{wo} - \lim_{n \rightarrow \omega} v_n^* x'_n v_n = P_L \hat{\pi}_\omega((x'_n)_n) P_L. \quad (2)$$

Let $y := \text{wo} - \lim_{n \rightarrow \omega} v_n^* x_n v_n$. Clearly $v_n^* x_n v_n \in P_L \mathbb{B}(H_\omega) P_L$, and for $\xi, \eta \in L$, we have

$$\begin{aligned} \langle y\xi, \eta \rangle &= \lim_{n \rightarrow \omega} \langle x_n v_n \xi, v_n \eta \rangle \\ &= \langle \pi_\omega((x_n)_n)(v_n \xi)_\omega, (v_n \eta)_\omega \rangle \\ &= \langle \pi_\omega((x_n)_n) \xi, \eta \rangle, \end{aligned}$$

whence (1) holds. (2) can be proved similarly.

Claim 2. The following holds:

$$P_L \pi_\omega(\ell^\infty(\mathbb{N}, M_n)) P_L \subset \pi_L(N), \quad (3)$$

$$P_L \hat{\pi}_\omega(\ell^\infty(\mathbb{N}, M'_n)) P_L \subset \pi_L(N)', \quad (4)$$

where π_L is the natural standard action of N on L unitarily equivalent to π_φ , i.e., $\pi_L(x) = w_0 \pi_\varphi(x) w_0^*$ ($x \in N$). By [AH12, Theorem 3.7, Corollary 3.28], $P_K = w w^*$, $w^*(\prod^\omega M_n) w = \widetilde{M}$ and $\rho: \widetilde{M} \ni \pi_{\tilde{\varphi}}(x) \mapsto w x w^* \in P_K(\prod^\omega M_n) P_K$ is a *-isomorphism. Note that by the existence of $\varepsilon: \widetilde{M} \rightarrow i(N)$, e satisfies $e \pi_{\tilde{\varphi}}((x_n)^\omega) e = \pi_\varphi(\varepsilon((x_n)^\omega)) e = e \pi_\varphi(\varepsilon((x_n)^\omega))$ for $(x_n)^\omega \in \widetilde{M}$. Now let $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$. Then by [AH12, Corollary 3.16], there exists $(a_n)_n \in \mathcal{M}^\omega(M_n, \varphi_n)$, $(b_n)_n \in \mathcal{L}_\omega(M_n, \varphi_n)$ and $(c_n)_n \in \mathcal{L}_\omega(M_n, \varphi_n)^*$ such that $x_n = a_n + b_n + c_n$ ($n \in \mathbb{N}$), and moreover

$$P_K \pi_\omega((x_n)_n) P_K = P_K (\pi_\omega((a_n)_n) P_K = w \pi_{\tilde{\varphi}}((a_n)^\omega) w^*.$$

Then as $P_L = e_L P_K$, it holds that

$$\begin{aligned} P_L \pi_\omega((x_n)_n) P_L &= e_L P_K \pi_\omega((x_n)_n) P_K e_L \\ &= e_L w \pi_{\tilde{\varphi}}((a_n)^\omega) w^* e_L \\ &= w_0 e \pi_{\tilde{\varphi}}((a_n)^\omega) e w_0^* \\ &= w_0 \pi_\varphi(\varepsilon((a_n)^\omega)) w_0^* \\ &= \pi_L(i^{-1} \circ \varepsilon((a_n)^\omega)) \in \pi_L(N), \end{aligned}$$

and (3) is proved. For (4), let $(x'_n)_n \in \ell^\infty(\mathbb{N}, M_n)$. Then $x'_n = J_n x_n J_n$, $x_n := J_n x'_n J_n \in M_n$ ($n \in \mathbb{N}$), and $\hat{\pi}_\omega((x'_n)_n) = J_\omega \pi_\omega((x_n)_n) J_\omega$. Since $P_K = p J_\omega p J_\omega$, where $p = \text{supp}((\varphi_n)_\omega) \in \prod^\omega M_n$, $P_K J_\omega = J_\omega P_K$ holds. Choose $(a_n)_n \in \mathcal{M}^\omega(M_n, \varphi_n)$ as above so that $w \pi_{\tilde{\varphi}}((a_n)^\omega) w^* = P_K \pi_\omega((x_n)_n) P_K$ holds. Thus,

it holds that

$$\begin{aligned}
P_L \hat{\pi}_\omega((x'_n)) P_L &= e_L P_K J_\omega \pi_\omega((x_n)_n) J_\omega P_K e_L \\
&= e_L J_\omega P_K \pi_\omega((x_n)_n) P_K J_\omega e_L \\
&= J_L e_L w \pi_{\tilde{\varphi}}((a_n)^\omega) w^* e_L J_L \\
&= J_L \pi_L(i^{-1} \circ \varepsilon((a_n)^\omega)) J_L \in \pi_L(N)',
\end{aligned}$$

and (4) is proved.

Claim 3. $\lim_{n \rightarrow \omega} v_n^* M_n v_n = \pi_L(N)$.

To show the claim, by [HW00, Lemma 5.2], it suffices to show the following:

$$\limsup_{n \rightarrow \omega} v_n^* M_n v_n \subset \pi_L(N), \quad (5)$$

$$\limsup_{n \rightarrow \omega} v_n^* M'_n v_n \subset \pi_L(N)'. \quad (6)$$

To prove (5) and (6), it is enough to show that for every $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ and $(x'_n)_n \in \ell^\infty(\mathbb{N}, M'_n)$,

$$\text{wo-} \lim_{n \rightarrow \omega} v_n^* x_n v_n \in \pi_L(N), \quad (7)$$

$$\text{wo-} \lim_{n \rightarrow \omega} v_n^* x'_n v_n \in \pi_L(N)'. \quad (8)$$

But they are the consequences of Claim 1 and Claim 2.

Finally, as the representation π_L is standard and $N \in \text{vN}^{\text{st}}(H)$, there exists $u \in \mathcal{U}(L, H)$ such that $u\pi_L(x)u^* = x$ ($x \in N$). Then by Claim 3, we have $u_n M u_n^* \rightarrow N$ in $\text{vN}(H)$, where $u_n = uv_n^* \in \mathcal{U}(H)$. From this and the fact that $\text{vN}(H)$ is a separable metrizable space, we may find a subsequence $\{n_k\}_{k=1}^\infty$ so that $u_{n_k} M u_{n_k}^* \rightarrow N$ ($k \rightarrow \infty$) in $\text{vN}(H)$. \square

In [HW00, Theorem 5.8], the second and the third-named authors proved that a type II_1 factor is in $\overline{\mathcal{F}_{\text{inj}}}$ (the closure of injective factors) if and only if it embeds into R^ω , where R is the injective II_1 factor. Let R_∞ (resp. R_λ) denote the injective factor of type III_1 (resp. type III_λ ($0 < \lambda < 1$)). As an application of the theorems above, we get the following result for general von Neumann algebras:

Theorem 3.4. *Let $N \in \text{vN}(H)$ and let $0 < \lambda < 1$. Then the following are equivalent:*

- (i) $N \in \overline{\mathcal{F}_{\text{inj}}}$.
- (ii) *There is an embedding $i: N \rightarrow R_\infty^\omega$ and a normal faithful conditional expectation $\varepsilon: R_\infty^\omega \rightarrow i(N)$.*
- (iii) *There is an embedding $i: N \rightarrow R_\lambda^\omega$ and a normal faithful conditional expectation $\varepsilon: R_\lambda^\omega \rightarrow i(N)$.*

Proof. (i) \Rightarrow (ii): By [HW00, Theorem 2.10 (ii)], $\mathcal{F}_{\text{III}_1} \cap \mathcal{F}_{\text{inj}}$ is dense in \mathcal{F}_{inj} . Therefore condition (i) implies that there exists a sequence $(M_n)_n$ of injective type III_1 factors on H such that $M_n \rightarrow N$. Then as $M_n \cong R_\infty$ ($n \in \mathbb{N}$), by Corollary 3.2, there exists a sequence $(\psi_n)_n \subset S_{\text{nf}}(R_\infty)$ such that there exists

an embedding $i: N \rightarrow (R_\infty, \psi_n)^\omega$ and a normal faithful conditional expectation $\varepsilon: (R_\infty, \psi_n)^\omega \rightarrow i(N)$. By [AH12, Theorem 6.11], we have $(R_\infty, \psi_n)^\omega \cong R_\infty^\omega$. Therefore (ii) holds. (ii) \Rightarrow (i): Let K_1, K_2 be Hilbert spaces such that $\tilde{N} := N \overline{\otimes} \mathbb{B}(K_1) \overline{\otimes} \mathbb{C}1_{K_2}$ acts standardly on $H \otimes K$, $K := K_1 \otimes K_2$. Then we obtain an embedding $i' = i \otimes \text{id} \otimes \text{id}: \tilde{N} \rightarrow Q^\omega$, $Q := R_\infty \overline{\otimes} \mathbb{B}(K_1) \overline{\otimes} \mathbb{C}1_{K_2}$ and a normal faithful conditional expectation $\varepsilon' = \varepsilon \otimes \text{id} \otimes \text{id}: Q^\omega \rightarrow i'(\tilde{N})$. Here we used the fact that $Q^\omega \cong R_\infty^\omega \overline{\otimes} \mathbb{B}(K_1) \overline{\otimes} \mathbb{C}1_{K_2}$ (cf. [MaTo12, Lemma 2.8]). Since $Q, \tilde{N} \in \text{vN}^{\text{st}}(H \otimes K)$, by Theorem 3.3, there exist $(w_n)_n \subset \mathcal{U}(H \otimes K)$ such that $w_n^* Q w_n \rightarrow \tilde{N}$ in $\text{vN}(H \otimes K)$. In particular, as $Q \cong R_\infty$, $\tilde{N} \in \overline{\mathcal{F}_{\text{inj}}}$ holds. Choose $v_0 \in \mathcal{U}(H \otimes K, H)$. Then by [HW00, Lemma 2.4], there exist $(u_n)_n \subset \mathcal{U}(H \otimes K)$ such that $v_0 u_n^* \tilde{N} u_n v_0^* \rightarrow N$ in $\text{vN}(H)$. As $v_0 u_n^* \tilde{N} u_n v_0^* \in \overline{\mathcal{F}_{\text{inj}}}$, this shows that $N \in \overline{\mathcal{F}_{\text{inj}}}$ holds. (i) \Leftrightarrow (iii) holds similarly. \square

Remark 3.5. Nou [Nou06] has shown that q -deformed Araki-Woods algebras (in the sense of Hiai [Hiai03], see also [Shl97]) have QWEP, whence by the above theorem they are embeddable into R_∞^ω within the range of a normal faithful conditional expectation.

The Effros-Maréchal topology gives an alternative proof of the following result without logician's method.

Corollary 3.6 (Farah-Hart-Sherman [FHS11]). *There exists a type II_1 factor M with separable predual such that every type II_1 factor N with separable predual admits an embedding $i: N \rightarrow M^\omega$.*

Proof. Let $\{M_n\}_{n=1}^\infty$ be a dense subset of $\mathcal{F}_{\text{II}_1}$. Define $M := \bigotimes_{n \in \mathbb{N}} (M_n, \tau_n)$. Then for every II_1 factor $N \in \text{vN}(H)$, there exists a sequence $\{n_k\}_{k=1}^\infty$ such that $M_{n_k} \rightarrow N$ ($k \rightarrow \infty$). Then as in the proof of Theorem 3.1, there is an embedding $i: N \rightarrow (M_{n_k}, \tau_{n_k})^\omega$ given by $i(x) = (x_k)^\omega$, where $(x_k)_k \in \ell^\infty(\mathbb{N}, M_{n_k})$ is such that $x_k \xrightarrow{\text{so}^*} x$. Since each M_{n_k} embeds into the II_1 factor M in a trace-preserving way, $(M_{n_k}, \tau_{n_k})^\omega$ embeds into the II_1 factor M^ω . Therefore N embeds into M^ω . \square

Raynaud showed [Ray02, Proposition 1.14] that $\prod^\mathcal{U} \mathbb{B}(H)$ is not semifinite where H is infinite-dimensional and \mathcal{U} is a free ultrafilter on the set I of all pairs (E, ε) where E is a finite-dimensional subspace of $\mathbb{B}(H)_*$ and $\varepsilon > 0$ partially ordered by $(E_1, \varepsilon_1) \leq (E_2, \varepsilon_2) \Leftrightarrow \varepsilon_1 \geq \varepsilon_2$ and $E_1 \subset E_2$. The proof is based on [Ray02, Lemma 1.12, Lemma 1.13] that $\mathbb{B}(H)^{**}$ is not semifinite, and by (local reflexivity) there is an embedding $i: \mathbb{B}(H)^{**} \rightarrow \prod^\mathcal{U} \mathbb{B}(H)$ and a normal faithful conditional expectation $\varepsilon: \prod^\mathcal{U} \mathbb{B}(H) \rightarrow i(\mathbb{B}(H)^{**})$. Therefore the large index set was required in his argument. We show that the same conclusion holds when I is replaced by \mathbb{N} using Effros-Maréchal topology.

Corollary 3.7. $\prod^\omega \mathbb{B}(H)$ is not semifinite.

Proof. Let $M \in \text{vN}(H)$ be an injective type III factor. Then by the proof of [HW00, Corollary 2.11], it is in the closure of the set $\mathcal{F}_{\text{I}_\infty}$ of type I_∞ factors on H . Then by Theorem 3.1, there is a sequence $\{\varphi_n\}_{n=1}^\infty \subset S_{\text{nf}}(\mathbb{B}(H))$, an embedding $i: M \rightarrow (\mathbb{B}(H), \varphi_n)^\omega$, and a normal faithful conditional expectation $\varepsilon: (\mathbb{B}(H), \varphi_n)^\omega \rightarrow i(M)$. In particular, $(\mathbb{B}(H), \varphi_n)^\omega$ is not semifinite by [Tom58,

Theorem 3]. Then by [AH12, Proposition 3.15], $\prod^\omega \mathbb{B}(H)$ has a non-semifinite corner $p(\prod^\omega \mathbb{B}(H))p \cong (\mathbb{B}(H), \varphi_n)^\omega$, $p = \text{supp}((\varphi_n)_\omega)$. \square

4 Characterizations of QWEP von Neumann Algebras

In this last section, we establish two characterizations of von Neumann algebras with QWEP. Recall first the definition of QWEP.

Definition 4.1. Let A be a C^* -algebra.

- (1) A is said to have the *weak expectation property* (WEP for short), if for any faithful representation $A \subset \mathbb{B}(H)$, there exists a unital completely positive (u.c.p. for short) map $\Phi: \mathbb{B}(H) \rightarrow A^{**}$ such that $\Phi(a) = a$ for all $a \in A$.
- (2) A is said to have the *quotient weak expectation property* (QWEP for short), if there exists a surjective $*$ -homomorphism from a C^* -algebra B with WEP onto A .

Next theorem characterizes QWEP von Neumann algebras in terms of Effros-Maréchal topology.

Theorem 4.2. Let H be a separable Hilbert space, and let $M \in \text{vN}(H)$. The following conditions are equivalent.

- (1) $M \in \overline{\mathcal{F}_{\text{inj}}}$.
- (2) M has QWEP.

Although it is not straightforward to prove that the Ocneanu ultraproduct of QWEP von Neumann algebras has QWEP by the definition, it is easy to do so by passing to the Groh-Raynaud ultraproduct.

Lemma 4.3. Let $\{(M_n, \varphi_n)\}_{n=1}^\infty$ be a sequence of σ -finite von Neumann algebras with faithful normal states. Assume that M_n has QWEP for each $n \in \mathbb{N}$. Then $(M_n, \varphi_n)^\omega$ has QWEP.

Proof. Since $(M_n, \varphi_n)^\omega \cong p(\prod^\omega M_n)p$, $p = \text{supp}((\varphi_n)_\omega)$, it suffices to show that $\prod^\omega M_n$ has QWEP. Since each M_n has QWEP, by [Kir93, Corollary 3.3 (i)], $\ell^\infty(\mathbb{N}, M_n)$ has QWEP, so does its quotient $(M_n)_\omega$. Therefore $((M_n)_\omega)^{**}$ has QWEP too [Kir93, Corollary 3.3 (v)]. Since $\prod^\omega M_n = \pi_\omega(\ell^\infty(\mathbb{N}, M_n))''$ is a corner of $((M_n)_\omega)^{**}$, it has QWEP. \square

Proof of Theorem 4.2. (1) \Rightarrow (2): Assume $M \in \overline{\mathcal{F}_{\text{inj}}}$. By Theorem 3.4, there is an embedding $i: M \rightarrow R_\infty^\omega$ with a normal faithful conditional expectation $\varepsilon: R_\infty^\omega \rightarrow i(M)$. By Lemma 4.3, R_∞^ω has QWEP, whence M has QWEP. (2) \Rightarrow (1): Let $M = \text{vN}(H)$ with QWEP. We reduce the problem to the case where von Neumann algebras involved are of finite type using the reduction method from [HJX10]. Consider $G = \mathbb{Z}[\frac{1}{2}]$ as a countable discrete subgroup of \mathbb{R} , and fix $\varphi \in S_{\text{nf}}(M)$. Consider the dual state $\widehat{\varphi}$ on $N := M \rtimes_{\sigma^\varphi} G$. There is a canonical embedding $\pi: M \rightarrow N$, and as G is countable discrete, there is a normal faithful conditional expectation $\varepsilon: N \rightarrow \pi(M)$. By [HJX10, Theorem

2.1], there exists an increasing sequence $\{N_n\}_{n=1}^\infty$ of finite von Neumann subalgebras of N with σso^* -dense union, together with a unique normal faithful conditional expectation $\varepsilon_n: N \rightarrow N_n$ ($n \in \mathbb{N}$) such that

$$\widehat{\varphi} \circ \varepsilon_n = \widehat{\varphi}, \quad \sigma_t^{\widehat{\varphi}} \circ \varepsilon_n = \varepsilon_n \circ \sigma_t^{\widehat{\varphi}}, \quad (t \in \mathbb{R}).$$

Moreover, it holds that (by [HJX10, Lemma 2.7])

$$\sigma\text{so}^* \text{-} \lim_{n \rightarrow \infty} \varepsilon_n(x) = x, \quad x \in N. \quad (9)$$

Let θ be the dual action of $\sigma^\varphi|_G$ on N . Then by Takesaki duality, N is isomorphic to the fixed point subalgebra of the QWEP algebra $N \rtimes_\theta \widehat{G} \cong M \overline{\otimes} \mathbb{B}(\ell^2(G))$ by the dual action of θ . Since $\widehat{\widehat{G}} = G$ is amenable, there is a conditional expectation of $N \rtimes_\theta \widehat{G}$ onto N , so N has QWEP as well. Again by the existence of ε_n , each N_n has QWEP too. Since N_n is of finite type, by [Kir93, Theorem 4.1], there is an embedding $i_n: N_n \rightarrow R^\omega$ (and a normal faithful conditional expectation $\varepsilon_n: R^\omega \rightarrow i_n(N_n)$, which automatically exists). Note also that there is an embedding i of R into $R \overline{\otimes} R_\infty \cong R_\infty$ and there is a normal faithful conditional expectation $\varepsilon: R \overline{\otimes} R_\infty \rightarrow R$ given by a right-slice map by some $\varphi \in S_{\text{nf}}(R_\infty)$. Let τ be the unique tracial state on R , and let $\psi := \tau \circ \varepsilon$. Then for any $(x_n)_n \in \ell^\infty(\mathbb{N}, R)$, $(i(x_n))_n$ is in $\mathcal{M}^\omega(R_\infty)$ (because $i(x_n) \in (R_\infty)_\psi$), whence we have an embedding $i^\omega: R^\omega \rightarrow R_\infty^\omega$, $(x_n)^\omega \mapsto (i(x_n))^\omega$, and a normal faithful conditional expectation $\varepsilon^\omega: R_\infty^\omega \rightarrow R_\infty$ given by $(x_n)^\omega \mapsto (\varepsilon(x_n))^\omega$. Therefore there is an embedding $i'_n = i^\omega \circ i_n: N_n \rightarrow R_\infty^\omega$ and a normal faithful conditional expectation $\varepsilon'_n = i^\omega \circ \varepsilon_n \circ (i^\omega)^{-1} \circ \varepsilon^\omega: R_\infty^\omega \rightarrow i'_n(N_n)$. Then by Theorem 3.4, $N_n \in \overline{\mathcal{F}_{\text{inj}}}$ holds. Now, as each N_n is in $\text{SA}_{\widehat{\varphi}}(N)$ and ε_n is the $\sigma^{\widehat{\varphi}}$ -preserving conditional expectation, the argument before [HW98, Corollary 2.12] gives $N_n \rightarrow N$ in $\text{vN}(H)$. This shows that $N \in \overline{\mathcal{F}_{\text{inj}}}$. Then by Theorem 3.4, there is an embedding $j: N \rightarrow R_\infty^\omega$ with a normal faithful conditional expectation $\varepsilon': R_\infty^\omega \rightarrow j(N)$. Therefore we obtain an embedding $i: M \xrightarrow{\pi} N \xrightarrow{j} R_\infty^\omega$ together with a normal faithful conditional expectation $R_\infty^\omega \xrightarrow{\varepsilon'} j(N) \xrightarrow{j \circ \varepsilon \circ j^{-1}} i(M)$. Again by Theorem 3.4, $M \in \overline{\mathcal{F}_{\text{inj}}}$ holds. \square

Remark 4.4. Although QWEP conjecture has not been settled, we remark that if there is at least one $M \in \text{vN}(H)$ without QWEP, then the set $\text{vN}(H)^{-\text{QWEP}}$ of those $M \in \text{vN}(H)$ without QWEP, is an open and dense subset of $\text{vN}(H)$. Indeed, by Theorem 4.2, $\text{vN}(H)^{-\text{QWEP}}$ is an open set. Suppose there is one $M \in \text{vN}(H)$ without QWEP, and let $N \in \text{vN}(H)$. Then by choosing $v_0 \in \mathcal{U}(H \otimes H, H)$, [HW00, Lemma 2.4], there is $(u_n)_n \in \mathcal{U}(H \otimes H)$ such that $v_0 u_n^* (N \overline{\otimes} M) u_n v_0^* \rightarrow N$ in $\text{vN}(H)$. Since $v_0 u_n^* (N \overline{\otimes} M) u_n v_0^*$ fails QWEP for each $n \in \mathbb{N}$, N is in the closure of $\text{vN}(H)^{-\text{QWEP}}$.

Theorem 4.5. *Let M be a von Neumann algebra with separable predual, and let \mathcal{P}_M^\natural be the natural cone in the standard form of M . The following conditions are equivalent.*

- (1) M has QWEP.
- (2) For any $n \in \mathbb{N}$, $\xi_1, \dots, \xi_n \in \mathcal{P}_M^\natural$, and $\varepsilon > 0$, there exists $k \in \mathbb{N}$ and $a_1, \dots, a_n \in M_k(\mathbb{C})_+$ such that

$$|\langle \xi_i, \xi_j \rangle - \text{tr}_k(a_i a_j)| < \varepsilon \quad (i, j = 1, \dots, n).$$

Here, tr_k is the normalized trace on $M_k(\mathbb{C})$.

Note that if M is a II_1 factor, then $\mathcal{P}_M^\natural = M_+ \xi_\tau$. Therefore the above theorem is analogous to the following Kirchberg's result [Kir93, Proposition 4.6] when unitaries in his results are replaced by positive elements.

Theorem 4.6 (Kirchberg). *Let M be a finite von Neumann algebra with separable predual with a normal faithful tracial state τ_M . Then the following conditions are equivalent:*

- (1) *There exists an embedding $i: M \rightarrow R^\omega$ with $\tau^\omega(i(a)) = \tau_M(a)$ ($a \in M$).*
- (2) *For every $\varepsilon > 0, n \in \mathbb{N}$ and $u_1, \dots, u_n \in \mathcal{U}(M)$, there is $k \in \mathbb{N}$ and $v_1, \dots, v_n \in \mathcal{U}(M_k(\mathbb{C}))$ such that*

$$|\tau_M(u_i^* u_j) - \text{tr}_k(v_i^* v_j)| < \varepsilon, \quad |\tau_M(u_i) - \text{tr}_k(v_i)| < \varepsilon$$

for $i, j = 1, \dots, n$.

The next lemma plays the key role in the proof of Theorem 4.5

Lemma 4.7. *Let $(M, H_M, J_M, \mathcal{P}_M^\natural), (N, H_N, J_N, \mathcal{P}_N^\natural)$ be standard forms. Assume that M has QWEP, N is σ -finite and there is a linear isometry $\rho: H_N \rightarrow H_M$ satisfying $\rho(\mathcal{P}_N^\natural) \subset \mathcal{P}_M^\natural$. Then N has QWEP.*

To prove the lemma, we use following classical results on the natural cone.

Lemma 4.8 (Araki). *Let $(M, H, J, \mathcal{P}_M^\natural)$ be a standard form with M σ -finite. Let $\varphi \in S_{\text{nf}}(M)$ with $\varphi = \omega_{\xi_\varphi}$, $\xi_\varphi \in \mathcal{P}_M^\natural$. Then $\Phi: M_{\text{sa}} \ni a \mapsto \Delta_\varphi^{\frac{1}{4}} a \xi_\varphi \in \mathcal{P}_M^\natural$ induces an order isomorphism between $\{a \in M_{\text{sa}}; -\alpha 1 \leq a \leq \alpha 1\}$ and $\{\xi \in H; J\xi = \xi, -\alpha \xi_\varphi \leq \xi \leq \alpha \xi_\varphi\}$ for each $\alpha > 0$.*

Proof. For $a \in M$, it is easy to see that

$$a \in M_+ \Leftrightarrow \Delta_\varphi^{\frac{1}{4}} a \xi_\varphi \in \mathcal{P}_M^\natural.$$

This implies that Φ is an order isomorphism from M_{sa} onto $\Phi(M_{\text{sa}})$. By [Ara74, Theorem 3.8 (8)], Φ maps $\{x \in M; 0 \leq x \leq \alpha 1\}$ onto $\{\eta \in \mathcal{P}_M^\natural; 0 \leq \eta \leq \alpha \xi_\varphi\}$. From this the claim easily follows. \square

Lemma 4.9. *Let $(M, H, J, \mathcal{P}^\natural)$ be a standard form. For each $\xi \in \mathcal{P}^\natural$, denote by $e(\xi)$ the support projection of ω_ξ . Let $q := e(\xi) J e(\xi) J$. Then the following holds.*

- (1) (Araki) $\xi \perp \eta \Leftrightarrow e(\xi) \perp e(\eta)$.
- (2) For $\xi, \eta \in \mathcal{P}^\natural$, $e(\xi) \leq e(\eta)$ if and only if $\eta \perp \zeta \Rightarrow \xi \perp \zeta$ holds for every $\zeta \in \mathcal{P}^\natural$.
- (3) $q(\mathcal{P}^\natural) = (\{\xi\}^\perp \cap \mathcal{P}^\natural)^\perp \cap \mathcal{P}^\natural$.

Proof. (1) (\Rightarrow) is [Ara74, Theorem 4, (7)], while (\Leftarrow) is clear. (2) and (3) have been known. We include a proof from second named author's master thesis for reader's convenience.

(2) (\Rightarrow) follows from (1). (\Leftarrow) Let $\{\xi_i\}_{i \in I}$ be a family of elements in \mathcal{P}_M^\natural such

that $\{e(\xi_i)\}_{i \in I}$ are pairwise orthogonal and $\sum_{i \in I} e(\xi_i) = 1 - e(\eta)$ (it can be proved that every σ -finite projection $p \in M$ is of the form $p = e(\eta)$ for some $\eta \in \mathcal{P}_M^\natural$). Hence $\xi_i \perp \eta$, which implies that $\xi_i \perp \xi$ for all $i \in I$. By (1), $e(\xi_i) \perp e(\xi)$ ($i \in I$) holds. Therefore

$$e(\xi) \leq 1 - \sum_{i \in I} e(\xi_i) = e(\eta).$$

(3) By (2), it holds that for $\eta \in \mathcal{P}_M^\natural$,

$$\begin{aligned} e(\eta) \leq e(\xi) &\Leftrightarrow \forall \zeta \in \mathcal{P}_M^\natural \ (\zeta \perp \xi \Rightarrow \zeta \perp \eta) \\ &\Leftrightarrow \eta \in (\{\xi\} \cap \mathcal{P}_M^\natural)^\perp \cap \mathcal{P}_M^\natural. \end{aligned}$$

If $e(\eta) \leq e(\xi)$, then $e(\xi)\eta = \eta$, so $q\eta = Je(\xi)Je(\xi)\eta = Je(\xi)\eta = J\eta = \eta$. Thus, $\eta = q(\eta) \in q(\mathcal{P}_M^\natural)$. On the other hand, if $\eta \in q(\mathcal{P}_M^\natural)$, then clearly $e(\eta) \leq e(\xi)$ holds. This shows that $q(\mathcal{P}_M^\natural) = (\{\xi\} \cap \mathcal{P}_M^\natural)^\perp \cap \mathcal{P}_M^\natural$. \square

The next result is [Oz04, Corollary 5.3].

Lemma 4.10 ([Kir93, Oz04]). *Let M be a von Neumann algebra, and if there is a C^* -algebra A with QWEP and a contractive linear map $\varphi: A \rightarrow M$ such that $\varphi(\overline{\text{Ball}}(A))$ is ultraweakly dense in $\overline{\text{Ball}}(M)$, then M has QWEP.*

Proof of Lemma 4.7. Put $e := \rho\rho^* \in \mathbb{B}(H_M)$.

Claim 1. $\rho(\mathcal{P}_N^\natural) = e(\mathcal{P}_M^\natural) = \mathcal{P}_M^\natural \cap e(H_M)$, and $\rho J_N = J_M \rho$.

First, $\rho(\mathcal{P}_N^\natural) = \rho\rho^*\rho(\mathcal{P}_N^\natural) \subset e(\mathcal{P}_M^\natural)$ holds. To prove $e(\mathcal{P}_M^\natural) \subset \rho(\mathcal{P}_N^\natural)$, we show $\rho^*(\mathcal{P}_M^\natural) \subset \mathcal{P}_N^\natural$. Let $\xi \in \mathcal{P}_M^\natural$. Then for each $\eta \in \mathcal{P}_N^\natural$, we have $\langle \rho^*(\xi), \eta \rangle = \langle \xi, \rho(\eta) \rangle \geq 0$, because $\rho(\eta) \in \mathcal{P}_M^\natural$, whence $\rho^*(\xi) \in (\mathcal{P}_M^\natural)^0 = \mathcal{P}_M^\natural$ and $\rho^*(\mathcal{P}_M^\natural) \subset \mathcal{P}_N^\natural$ holds. Therefore for $\xi \in \mathcal{P}_M^\natural$, $e\xi = \rho(\rho^*\xi) \in \rho\mathcal{P}_N^\natural$. This proves $\rho(\mathcal{P}_N^\natural) = e\mathcal{P}_M^\natural$. Note that this also shows that $e(\mathcal{P}_M^\natural) \subset \mathcal{P}_M^\natural$. Therefore $e(\mathcal{P}_M^\natural) = \mathcal{P}_M^\natural \cap e(H_M)$ holds too. To see that last equality, recall that any $\xi \in H_N$ can be written as $xi = (\xi_1 - \xi_2) + i(\xi_3 - \xi_4)$, where $\xi_j \in \mathcal{P}_N^\natural$ and $J_N \xi_j = \xi_j$ ($1 \leq j \leq 4$). From this it holds that

$$\begin{aligned} \rho J_N \xi &= \rho \{(\xi_1 - \xi_2) - i(\xi_3 - \xi_4)\} = J_M \{(\rho\xi_1 - \rho\xi_2) + i(\rho\xi_3 - \rho\xi_4)\} \\ &= J_M \rho\xi. \end{aligned}$$

Now fix $\psi \in S_{\text{nf}}(N)$. Then $\psi = \omega_{\xi_\psi}, \xi_\psi \in \mathcal{P}_N^\natural$. Let $\varphi := \omega_\eta, \eta := \rho(\xi_\psi) \in \mathcal{P}_M^\natural$, and $p := \text{supp}(\varphi) \in M$.

Claim 2. $q := pJ_M pJ_M \geq e$.

Since H_M is spanned by \mathcal{P}_M^\natural , it suffices to show that $q(\mathcal{P}_M^\natural) \supset e(\mathcal{P}_M^\natural) = \rho(\mathcal{P}_N^\natural)$. Let $\xi \in \mathcal{P}_N^\natural$. Let $\zeta \in \{\rho(\xi_\psi)\}^\perp \cap \mathcal{P}_M^\natural$. Then $0 = \langle \zeta, \rho(\xi_\psi) \rangle = \langle \rho^*(\zeta), \xi_\psi \rangle$. Since $\psi = \omega_{\xi_\psi}$ is faithful, by Lemma 4.9 (1), $\zeta \perp \xi_\psi \Rightarrow \zeta \perp \xi$ holds for every $\zeta \in \mathcal{P}_N^\natural$. Therefore $\rho^*(\zeta) \perp \xi \Leftrightarrow \zeta \perp \rho(\xi)$ holds. This implies, by Lemma 4.9 (3), that

$$\rho(\xi) \in (\{\rho(\xi_\psi)\}^\perp \cap \mathcal{P}_M^\natural)^\perp \cap \mathcal{P}_M^\natural = q(\mathcal{P}_M^\natural).$$

Therefore $e(\mathcal{P}_M^\natural) = \rho(\mathcal{P}_N^\natural) \subset q(\mathcal{P}_M^\natural)$, and $e \leq q$ holds.

Since M has QWEP, so does qMq . Therefore we may replace $(M, H_M, J_M, \mathcal{P}_M^\natural)$ by $(qMq, qH_M, J_M|_{qH_M}, q(\mathcal{P}_M^\natural))$ and assume that $q = 1$, and φ is faithful on M with $\xi_\varphi = \rho(\xi_\psi) \in \mathcal{P}_M^\natural$. In this case we may identify $\mathcal{P}_M^\natural = \Delta_\varphi^{\frac{1}{4}} M_+ \xi_\varphi$, $\mathcal{P}_N^\natural = \Delta_\psi^{\frac{1}{4}} N_+ \xi_\psi$.

Claim 3. There are unital positive maps $i: N \rightarrow M, \varepsilon: M \rightarrow N$ such that the following diagram commutes.

$$\begin{array}{ccccc} N_+ & \xrightarrow{i} & M_+ & \xrightarrow{\varepsilon} & N_+ \\ \beta \downarrow & & \downarrow \alpha & & \downarrow \beta \\ \mathcal{P}_N^\natural & \xrightarrow{\rho} & \mathcal{P}_M^\natural & \xrightarrow{\rho^*} & \mathcal{P}_N^\natural \end{array}$$

where $\alpha: M_+ \ni x \mapsto \Delta_\varphi^{\frac{1}{4}} x \xi_\varphi \in \mathcal{P}_M^\natural, \beta: N_+ \ni y \mapsto \Delta_\psi^{\frac{1}{4}} y \xi_\psi \in \mathcal{P}_N^\natural$. To construct i , let $y \in N_+$. Then as $0 \leq y \leq \|y\|1$, Lemma 4.8 asserts that $\Delta_\psi^{\frac{1}{4}} y \xi_\psi \leq \|y\| \xi_\psi$ in \mathcal{P}_N^\natural . Therefore $\rho(\Delta_\psi^{\frac{1}{4}} y \xi_\psi) \leq \rho(\|y\| \xi_\psi) = \|y\| \xi_\varphi$ in \mathcal{P}_M^\natural holds. Again by Lemma 4.8 applied to \mathcal{P}_M^\natural , there exists unique $\theta(y) \in M_+, 0 \leq \|\theta(y)\| \leq \|y\|$, such that $\rho(\Delta_\psi^{\frac{1}{4}} y \xi_\psi) = \Delta_\varphi^{\frac{1}{4}} \theta(y) \xi_\varphi$. It is clear from the construction that $N_+ \ni y \mapsto \theta(y) \in M_+$ is an affine map. Let now $y \in N$. There exist unique $y_k \in N_+ (1 \leq k \leq 4)$ with $y_1 y_2 = y_2 y_1 = 0 = y_3 y_4 = y_4 y_3$ such that $y = (y_1 - y_2) + i(y_3 - y_4)$. Then define

$$i(y) := (\theta(y_1) - \theta(y_2)) + i(\theta(y_3) - \theta(y_4)).$$

Then $i: N \rightarrow M$ is a positive linear map. Similarly, we define ε by

$$\varepsilon(x) := (\pi(x_1) - \pi(x_2)) + i(\pi(x_3) - \pi(x_4)),$$

for $x = (x_1 - x_2) + i(x_3 - x_4)$ with $x_k \in M_+ (1 \leq k \leq 4), x_1 x_2 = x_2 x_1 = 0 = x_3 x_4 = x_4 x_3$, where for $x \in M_+, \pi(x) \in N_+$ is the unique element satisfying

$$\rho^*(\Delta_\varphi^{\frac{1}{4}} x \xi_\varphi) = \Delta_\psi^{\frac{1}{4}} \pi(x) \xi_\psi.$$

Note that since $\rho^*(\mathcal{P}_M^\natural) \subset \mathcal{P}_N^\natural, \pi(x)$ is well-defined by Lemma 4.8. By construction, the above diagram commutes. Finally, let $y \in N_+$. Then by construction, $\rho(\Delta_\psi^{\frac{1}{4}} y \xi_\psi) = \Delta_\varphi^{\frac{1}{4}} i(y) \xi_\varphi$ holds. Then we apply ρ^* to obtain $\Delta_\psi^{\frac{1}{4}} y \xi_\psi = \rho^*(\Delta_\varphi^{\frac{1}{4}} i(y) \xi_\varphi)$. Therefore by the construction of $\varepsilon, \varepsilon \circ i(y) = y$ holds. Therefore by linearity, $\varepsilon \circ i = \text{id}_N$. In particular, ε maps the closed unit ball of M onto the closed unit ball of N . Since ε is positive hence contractive, and since M has QWEP, N also has QWEP by Lemma 4.10. \square

Proof of Theorem 4.5. (1) \Rightarrow (2) Assume that M has QWEP. then by Theorem 4.2, $M \in \overline{\mathcal{F}_{\text{inj}}}$ holds. By (the proof of) [HW00, Corollary 2.11], the set \mathcal{F}_{fin} of finite type I factors is dense in \mathcal{F}_{inj} . Therefore there is $\{k_n\}_{n=1}^\infty \subset \mathbb{N}$ and $N_n \in \text{vN}(H)$ with $N_n \cong M_{k_n}(\mathbb{C}) (n \in \mathbb{N})$ such that $N_n \rightarrow M$ in $\text{vN}(H)$. Then by Theorem 3.1, there is $\varphi_n \in S_{\text{nf}}(M_{k_n}(\mathbb{C})) (n \in \mathbb{N})$, an embedding $i: M \rightarrow (M_{k_n}(\mathbb{C}), \varphi_n)^\omega$, and a normal faithful conditional expectation $\varepsilon: (M_{k_n}(\mathbb{C}), \varphi_n)^\omega \rightarrow i(M)$. Let $(M_{k_n}(\mathbb{C}), K_n, J_n, \mathcal{P}_n^\natural)$ be the standard

form constructed from the GNS representation of φ_n for each $n \in \mathbb{N}$. Then by [AH12, Theorem 3.18], $(\prod^\omega M_{k_n}(\mathbb{C}), K_\omega, J_\omega, \mathcal{P}_\omega^\natural)$ is a standard form, where $(K_\omega, J_\omega, \mathcal{P}_\omega^\natural)$ is the ultraproduct of $(K_n, J_n, \mathcal{P}_n^\natural)_n$ (see [AH12, Theorem 3.18]). Let $p = \text{supp}((\varphi_n)_\omega) \in \prod^\omega M_{k_n}(\mathbb{C})$, and $q := pJ_\omega pJ_\omega$. Then by [AH12, Corollaries 3.27 and 3.28], $(q(\prod^\omega M_{k_n}(\mathbb{C}))q, qK_\omega, J_\omega|_{qK_\omega}, q\mathcal{P}_\omega^\natural)$ can be regarded as a standard representation of $\widetilde{M} = (M_{k_n}(\mathbb{C}), \varphi_n)^\omega$. Recall also that the natural cone \mathcal{P}_n^\natural can be also regarded as the one constructed from the GNS representation of the trace tr_{k_n} , we see that \mathcal{P}_n^\natural satisfies condition (2) for each n . Then $\mathcal{P}_\omega^\natural$ also satisfies condition (2): let $\varepsilon > 0, m \in \mathbb{N}$ and $\xi_1 = (\xi_{1,n})_\omega, \dots, \xi_m = (\xi_{m,n})_\omega \in \mathcal{P}_\omega^\natural$ be given, where $\xi_{i,n} \in \mathcal{P}_n^\natural$ ($1 \leq i \leq m, n \in \mathbb{N}$). Then we have

$$\langle \xi_i, \xi_j \rangle = \lim_{n \rightarrow \omega} \langle \xi_{i,n}, \xi_{j,n} \rangle = \lim_{n \rightarrow \omega} \text{tr}_{k_n}(\xi_{i,n} \xi_{j,n}) \quad (1 \leq i, j \leq m),$$

so that

$$J := \{n \in \mathbb{N}; |\langle \xi_i, \xi_j \rangle - \text{tr}_{k_n}(\xi_{i,n} \xi_{j,n})| < \varepsilon, 1 \leq i, j \leq m\} \in \omega.$$

Choose $n_0 \in J$, and put $\eta_i := \xi_{i,n_0}$ ($1 \leq i \leq m$). Then the inequality in (2) is satisfied. But as $q(\mathcal{P}_\omega^\natural) \subset \mathcal{P}_\omega^\natural$, $q(\mathcal{P}_\omega^\natural)$ also satisfies (2). Furthermore, $i(M) \subset \widetilde{M}$ and we have a normal faithful conditional expectation $\varepsilon: \widetilde{M} \rightarrow i(M)$. Thus choosing $\psi \in S_{\text{nf}}(i(M))$ and letting $\widetilde{\psi} := \psi \circ \varepsilon$, we obtain

$$\mathcal{P}_M^\natural \cong \mathcal{P}_{i(M)}^\natural = \overline{\Delta_\psi^{\frac{1}{4}} i(M)_+ \xi_\psi} \subset \mathcal{P}_{\widetilde{M}}^\natural = \overline{\Delta_{\widetilde{\psi}}^{\frac{1}{4}} \widetilde{M}_+ \xi_{\widetilde{\psi}}} \cong q(\mathcal{P}_\omega^\natural).$$

Here, \cong means that the one natural cone is mapped to the other by a unitary implementing the isomorphism of a standard form. Therefore \mathcal{P}_M^\natural satisfies condition (2).

(2) \Rightarrow (1) Assume M satisfies (2). Let $\{\xi_n\}_{n=1}^\infty \subset \mathcal{P}_M^\natural$ be a dense sequence. Using condition (2), for each $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ and $a_1^{(n)}, \dots, a_n^{(n)} \in M_{k_n}(\mathbb{C})_+$ such that

$$|\langle \xi_i, \xi_j \rangle - \text{tr}_{k_n}(a_i^{(n)} a_j^{(n)})| < \frac{1}{n}, \quad 1 \leq i, j \leq n.$$

Put

$$\xi_i := \begin{cases} a_i^{(n)} \xi_{\text{tr}_{k_n}} \in \mathcal{P}_n^\natural := \mathcal{P}_{M_{k_n}(\mathbb{C})}^\natural & (n \geq i) \\ 0 & (n < i) \end{cases},$$

and $\widetilde{\xi}_i := (\xi_i^{(n)})_\omega \in \mathcal{P}_\omega^\natural := (\mathcal{P}_n^\natural)_\omega$. Then

$$\langle \widetilde{\xi}_i, \widetilde{\xi}_j \rangle = \lim_{n \rightarrow \omega} \langle \xi_i^{(n)}, \xi_j^{(n)} \rangle = \langle \xi_i, \xi_j \rangle, \quad i, j \in \mathbb{N}.$$

Fix one $\varphi \in \text{nf}(M)$. Then on the dense subspace $K_0 := \text{span}\{\xi_i; i \geq 1\}$ of $L^2(M, \varphi)$, define $\rho_0: K_0 \rightarrow H_\omega$ by

$$\rho_0 \left(\sum_{j=1}^k \lambda_{ij} \xi_{ij} \right) := \sum_{j=1}^k \lambda_{ij} \widetilde{\xi}_{ij}, \quad \lambda_{ij} \in \mathbb{C} \quad (1 \leq i \leq k).$$

Then ρ_0 is uniquely extended to a linear isometry $\rho: L^2(M, \varphi) \rightarrow H_\omega$ such that $\rho(\mathcal{P}_M^\natural) \subset \mathcal{P}_\omega^\natural$. Since $\widetilde{M} := \prod^\omega M_{k_n}(\mathbb{C})$ has QWEP with natural cone $\mathcal{P}_\omega^\natural$, M has QWEP thanks to Lemma 4.7. \square

As a corollary, we have the following:

Theorem 4.11. *Let $0 < \lambda < 1$. The following conditions are equivalent:*

- (a) \mathcal{F}_{inj} is dense in $\text{vN}(H)$.
- (b) For any $N \in \text{vN}(H)$, there is an embedding $i: N \rightarrow R_\infty^\omega$ and a normal faithful conditional expectation $\varepsilon: R_\infty^\omega \rightarrow i(N)$.
- (c) For any $N \in \text{vN}(H)$, there is an embedding $i: N \rightarrow R_\lambda^\omega$ and a normal faithful conditional expectation $\varepsilon: R_\lambda^\omega \rightarrow i(N)$.
- (d) For any $N \in \mathcal{F}_{\text{II}_1}$, there is an embedding $i: N \rightarrow R^\omega$.
- (e) Every $N \in \text{vN}(H)$ has QWEP.
- (f) For any $N \in \text{vN}(H)$, there is $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$, a sequence $\varphi_k \in S_{\text{nf}}(M_{n_k}(\mathbb{C}))$ such that N admits an embedding $i: N \rightarrow (M_{n_k}(\mathbb{C}), \varphi_k)^\omega$ and a normal faithful conditional expectation $\varepsilon: (M_{n_k}(\mathbb{C}), \varphi_k)^\omega \rightarrow i(N)$.

Acknowledgements

The authors thank Masaki Izumi for useful comments and informations about literature. HA is supported by EPDI/JSPS/IHÉS Fellowship. UH is supported by ERC Advanced Grant No. OAFPG 27731 and the Danish National Research Foundation through the Center for Symmetry and Deformation.

Appendix

In this appendix we add proofs for the following lemmata. Both of them have been known, and the latter is essentially due to Araki [Ara74]. We include proofs for reader's convenience.

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